5.2 - Linear Models: Boundary-Value Problems

Consider $y'' + 3y = 0,\ \underbrace{y(0) = 0,}\ \underbrace{y(\pi) = 0}$

boundary conditions

This has the trivial solution $y=0$.

We can find that $y = a\cos(\sqrt{3}x) + b\sin(\sqrt{3}x)$

since $y(0)=0$, $a=0$

$y(\pi)=0 \Rightarrow b\sin(\sqrt{3}\pi)=0 \Rightarrow b=0$

We only have the trivial solution.

Now consider $y'' + 4y = 0,\ y(0) = 0,\ y(\pi) = 0$

$\longrightarrow a=0$

$y = a\cos 2x + b\sin 2x$

$y = b\sin 2x$ with $y(\pi)=0$

$0 = b\sin 2\pi$ holds true $\forall$ b $\in \mathbb{R}$. (for all, in)

Then $y = b\sin 2x$ is a solution to the BVP for $b \in \mathbb{R}$. The BVP has an infinite number of solutions.

Definition: A number λ for which there exist nontrivial solutions to $y'' + P(x)y' + \lambda Q(x)y = 0,\ y(a) = 0,\ y(b) = 0$ is called an **eigenvalue**, and a solution $y = ay_1 + by_2$ that satisfies the BVP is an **eigenfunction** associated with the eigenvalue λ.

In the previous example, $P(x)=0$, $Q(x)=1$

$\lambda=4$ is the eigenvalue, and $y=b\sin 2x$

is the eigenfunction.

$y''+3y=0$ has no eigenvalue.

We can generalize this:

Example: Determine eigenvalues and eigenfunctions for
$y''+\lambda y=0,\ y(0)=0,\ y(L)=0\ (L>0)$.

$m^2=0 \Rightarrow m=0$

Case I: $\lambda=0$. Then $y=ax+b$

and the boundary conditions give $a=b=0$

$y=0$ the trivial solution - no eigenvalue

Case II: $\lambda<0$. Now let $\lambda=-\alpha^2$

so $y''-\alpha^2 y=0 \Rightarrow m^2-\alpha^2=0 \Rightarrow m=\pm\alpha$

$y=ae^{\alpha x}+be^{-\alpha x}$

Recall $\cosh x=\dfrac{e^x+e^{-x}}{2}$, $\sinh x=\dfrac{e^x-e^{-x}}{2}$

so we can write $y=a\cosh\alpha x+b\sinh\alpha x$

$y(0)=0 \Rightarrow a=0$, $y(L)=0 \Rightarrow b=0$

Again, only the trivial solution - no eigenvalue.

Case III: $\lambda>0$, say $\lambda=\alpha^2$

$y''+\alpha^2 y=0 \Rightarrow m=\pm\alpha i$

$$y = a\cos\alpha x + b\sin\alpha x$$

$y(0) = 0 \Rightarrow a = 0$, and $y(L) = 0$ yields

$$b\sin\alpha L = 0$$

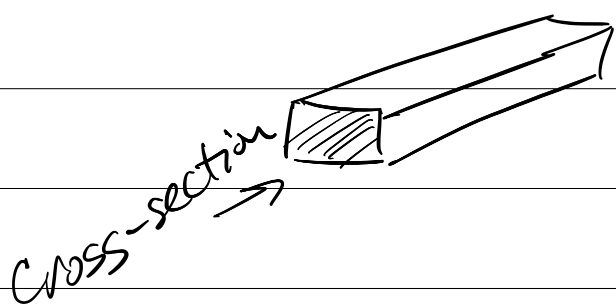
this is zero whenever $\alpha L = n\pi$, $n \in \mathbb{Z}^+$

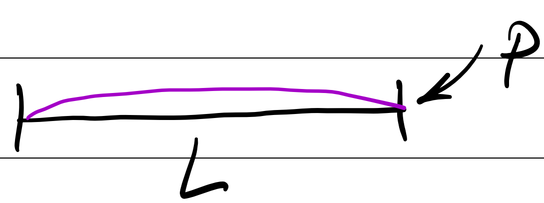
Assuming $\alpha > 0$ and $L > 0$,

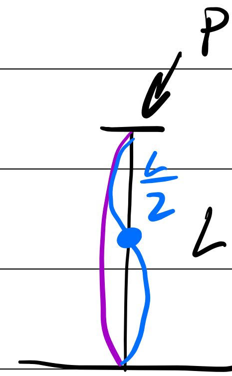
$$\alpha = \frac{n\pi}{L} \Rightarrow \lambda = \alpha^2 = \frac{n^2\pi^2}{L^2}$$

we have an infinite sequence of eigenvalues:

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad n = 1, 2, 3, \ldots$$







Consider a vertical beam of length L with load P applied to one end or a road with compressive force P applied to one end. Theory of elasticity provides the differential equation $EI\dfrac{d^2y}{dx^2} = -Py$ or $EI\dfrac{d^2y}{dx^2} + Py = 0$ which in standard form is

$$\frac{d^2y}{dx^2} + \frac{P}{EI}y = 0$$

The product EI is the flexural rigidity of the beam, E is Young's modulus of elasticity, and I is the moment of inertia of a cross-section of the beam (the shape's inherent resistance to bending).

Considering $P(x) = 0,\ Q(x) = 1$, and letting $\lambda = \dfrac{P}{EI}$, this has the form

$$y'' + \lambda y = 0$$

With this substitution and considering $y(0)=0$, $y(L)=0$, we have solutions

$$y = b\sin\left(\frac{n\pi x}{L}\right)$$

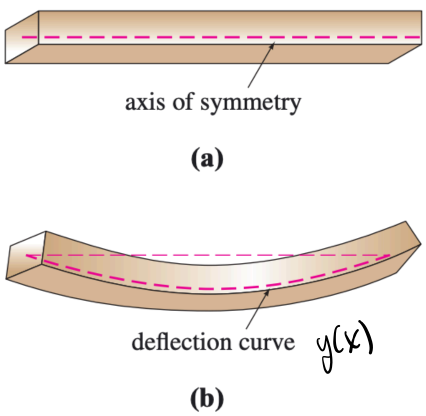
so the column/road will buckle/deflect if $P = \dfrac{n^2\pi^2 EI}{L^2}$. When $n=1$, we have the first buckling load.

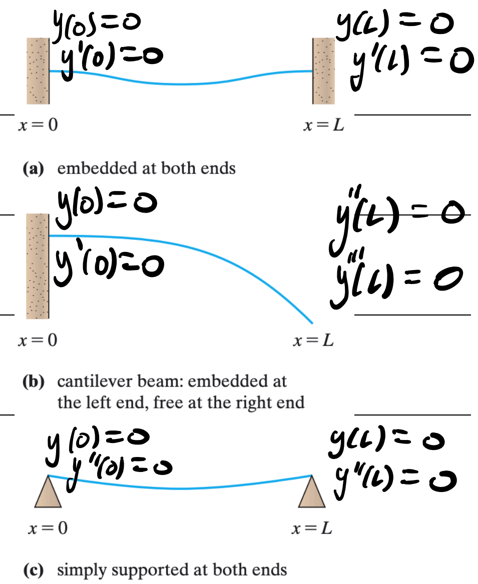
If we restrain/support the system at $\frac{L}{2}$, the smallest critical load (buckling load) will be $P = \dfrac{4\pi^2 EI}{L^2}$.

Considering a beam of length L, the bending moment (response or resistance to load) $M(x)$ is related to the load per unit length $w(x)$ by $\frac{d^2M}{dx^2} = w(x)$ (this is from theory of elasticity). Assuming down is positive, $M(x) = EI\kappa$ is proportional to the curvature of the elastic curve. From multivariable calculus, curvature is $\kappa = \frac{y''}{[1+(y')^2]^{3/2}}$. For small deflection, $y' = 0$, so $M = EIy''$. We see that $M'' = w(x)$ becomes

$$EI\frac{d^4y}{dx^4} = w(x)$$

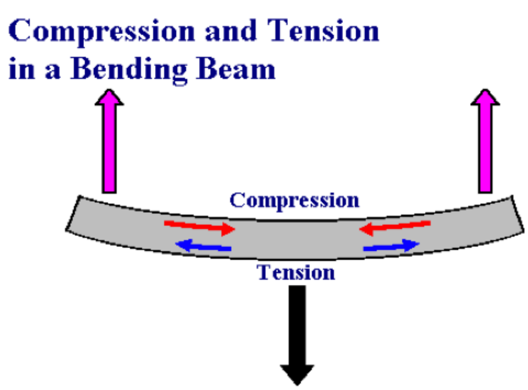
Definition: The graph of the function $y(x)$ is the **deflection curve** of the beam.





For the deflection $y(x)$, we have the following:
y' is the slope of the curve;
y'' is the bending moment;
y''' is the shear force.

If the end of the beam is
embedded, $y=0$, $y'=0$
free, $y''=0$, $y'''=0$
simply supported/hinged
$y=0$, $y''=0$



Example: Solve the differential equation $EI\dfrac{d^4y}{dx^4} = w(x)$ subject to the appropriate boundary conditions. The beam is of length L, and w_o is a constant.

(a) The beam is simply supported at both ends, and
$w(x) = w_0,\ 0 < x < L.$

$$\frac{d^4y}{dx^4} = \frac{w_0}{EI} \Rightarrow y''' = \frac{w_0}{EI}x + C_1$$

$$y'' = \frac{w_0}{2EI}x^2 + C_1x + \cancel{C_2}$$

$$y' = \frac{w_0}{6EI}x^3 + \frac{1}{2}C_1x^2 + \cancel{C_2x} + C_3$$

$$y = \frac{w_0}{24EI}x^4 + \frac{1}{6}C_1x^3 + \cancel{\frac{1}{2}C_2x^2} + C_3x + \cancel{C_4}$$

$$y(0) = y''(0) = y(L) = y''(L) = 0$$

$\hookrightarrow C_4 = 0$ $\quad\hookrightarrow C_2 = 0$

$y''(L) = 0 \Rightarrow 0 = \dfrac{w_0}{2EI}L^2 + C_1L$ gives C_1

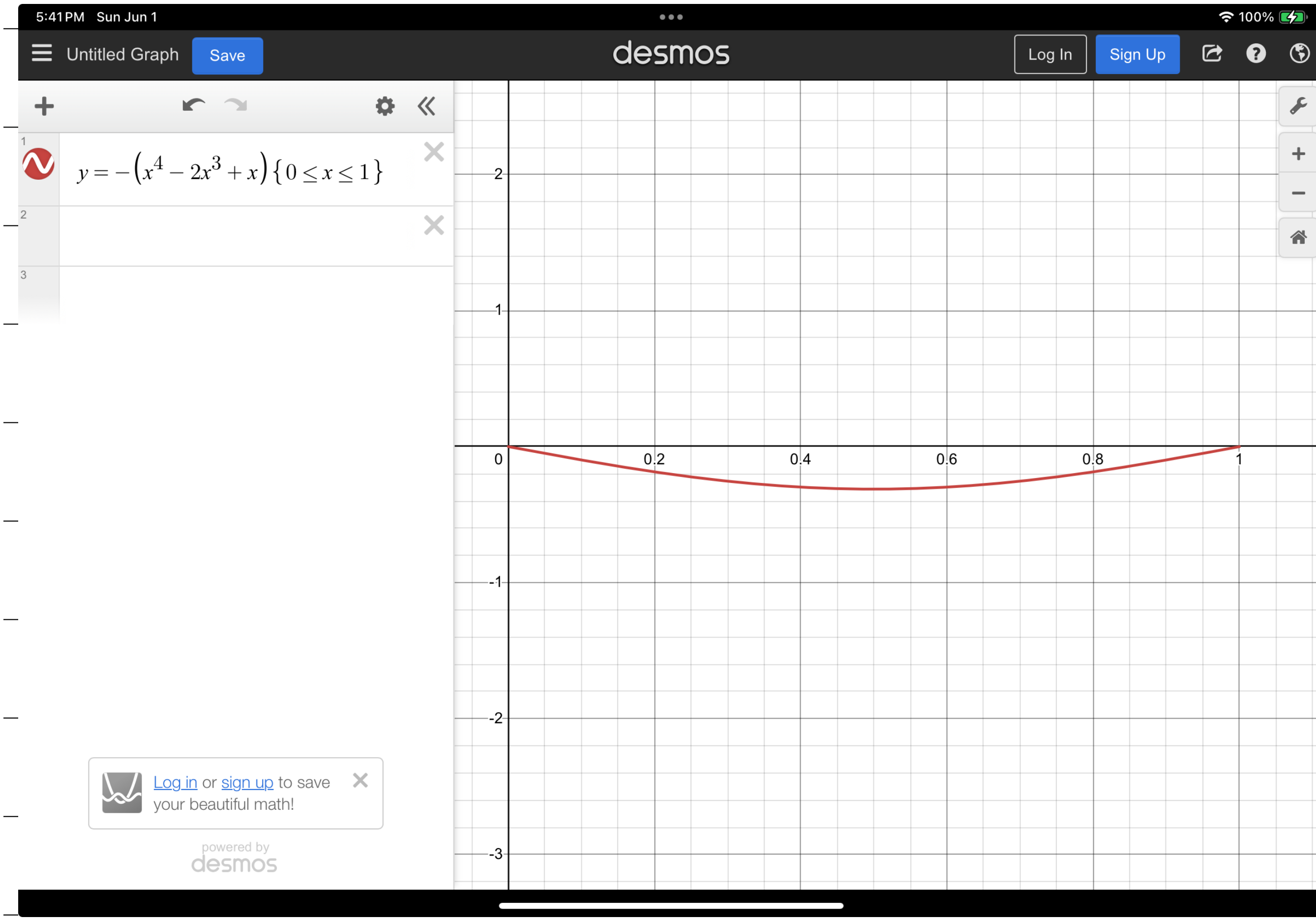
C_1, with $y(L) = 0$ gives C_3

We find that $y(x) = \dfrac{w_0}{24EI}\left(x^4 - 2Lx^3 + L^3x\right)$

(b) Use a graphing utility to graph the deflection curve when
$w_0 = 24EI$ and $L = 1$.

$$y(x) = x^4 - 2x^3 + x, \quad 0 \le x \le 1$$

Graph $-y(x) = -(x^4 - 2x^3 + x)$



$$x = \int [(D+1)y - 6]\,dt + c_3$$

$$Dx + (D+1)y = -6$$

4.9 : ① $(2D+1)(D-1)x - (2D+1)y = 6$

② $(D-1)x + Dy = -6$

① $-(2D+1)$②

$$(2D+1)(D-1)x - (2D+1)y = 6$$

$$-\cancel{(2D+1)(D-1)x} - (2D^2+D)y = 6$$

$$(2D^2+3D+1)y = -12$$

$$(2m+1)(m+1) = 0 \Rightarrow m = -\tfrac{1}{2}, -1$$

$$y_c = c_1 e^{-1/2x} + c_2 e^{-x} \quad , \quad y_p = A \Rightarrow A = -12$$

$$y = c_1 e^{-1/2} + c_2 e^{-x} - 12$$

$$(2D+1)(D-1)x - (2D+1)y = 6$$

$$(D-1)x + Dy = -6$$

D① $+ (2D+1)$②

$$D(2D+1)(D-1)x$$

$$(2D^3 - D^2 - D)x - \cancel{D(2D+1)y} = 0$$

$$(2D+1)(D-1)x$$

$$(2D^2 - D - 1)x + \cancel{D(2D+1)y} = -6$$

$$(2D^3 + D^2 - 2D - 1)x = -6$$

$$2m^3+m^2-2m-1=0$$

$$(m^2-1)(2m+1)=0 \Rightarrow m=\pm 1, -\frac{1}{2}$$

$$x_c = c_3e^{t}+c_4e^{-t}+c_5e^{-1/2t} \qquad y_p = A \Rightarrow A=6$$

$$x(t) = c_3e^{t}+c_4e^{-t}+c_5e^{-1/2t}+6$$

$$y(t) = c_1e^{-1/2t}+c_2e^{-t}-12$$

$$Dy = -\frac{1}{2}c_1e^{-1/2t}-c_2e^{-t}$$

$$(2D+1)(D-1)x-(2D+1)y=6$$

$$(D-1)x+Dy=-6$$

$$Dx = c_3e^{t}-c_4e^{-t}-\frac{1}{2}c_5e^{-1/2t}$$

$$-2c_4e^{-t}-\frac{3}{2}c_5e^{-1/2t}-\cancel{6}-\frac{1}{2}c_1e^{-1/2t}-c_2e^{-t}=-\cancel{6}$$

$$-2c_4-c_2=0 \qquad -\frac{3}{2}c_5-\frac{1}{2}c_1=0$$

$$c_4=-\frac{1}{2}c_2 \qquad c_5=-\frac{1}{3}c_1$$

$$\boxed{\begin{aligned} x(t)&=c_3e^{t}-\frac{1}{2}c_2e^{-t}-\frac{1}{3}c_1e^{-1/2t}+6 \\ y(t)&=c_1e^{-1/2t}+c_2e^{-t}+12 \end{aligned}}$$

$$\frac{dx}{dt} = -y + t \qquad Dx + y = t$$

$$\frac{dy}{dt} = x - t \quad \to \quad -D(x - Dy) = -D(t)$$

$$(D^2+1)y = t - 1 \qquad m = \pm i$$

$y_c = c_1 \cos t + c_2 \sin t$, $\quad y_p = At + B$

$A = 1$, $B = -1$

$y = c_1 \cos t + c_2 \sin t + t - 1 \longrightarrow$ ① $Dx = -y + t$

$$\cancel{Dx}\; x = \int (-c_1 \cos t - c_2 \sin t + 1)\, dt$$

$$x = -c_1 \sin t + c_2 \cos t + t + c_3$$

$x - Dy = t$: $\quad -c_1 \cancel{\sin t} + c_2 \cancel{\cos t} + t + c_3$

$\quad + c_1 \cancel{\sin t} - c_2 \cancel{\cos t} - 1 \qquad = t$

$c_3 = 1$

$$\boxed{\begin{aligned} x &= -c_1 \sin t + c_2 \cos t + t + 1 \\ y &= c_1 \cos t + c_2 \sin t + t - 1 \end{aligned}}$$